A convex optimization model for finding non-negative polynomials ☆

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Abstract

This paper presents a convex optimization model for the problem of finding some polynomials for which certain linear combinations are non-negative polynomials. This model is then applied to solve several filter design problems. We first reformulate some low-pass filter design problems, with finite or infinite impulse response, as optimization problems over non-negative (real or complex) polynomials whose feasibility problems can be solved by applying our model. The whole optimization problems are then solved by using a combination of a bisection search procedure on an appropriate parameter and our convex optimization model to solve the feasibility problems. Some numerical examples illustrate the method.

Keywords:
Non-negative polynomial, sum of squares, sum of square magnitudes of polynomials, filter design problem.

1. Motivation

Filter design problems are particularly important in linear time-invariant systems theory. Many designs of filters and methods to solve the corresponding problems have

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been proposed in the literature, see, e.g., [1, Chapter 7] for a wide range of methods for designing finite or infinite impulse response filters. We will write FIR filter or IIR filter if the filter is a finite or infinite impulse response system, respectively. In the papers of Genin et al. [2] and Wu et al. [3], it is shown that the IIR or FIR filter design problems can be reformulated as optimization problems over non-negative polynomials. More precisely, the feasibility problems of such optimization problems require to find some polynomials such that some linear combinations of them are non-negative univariate polynomials.

Even though the whole optimization problem is not convex, in [4] the authors show that the feasibility problem can be reformulated as a conic linear programming [5] (conic LP) problem over the convex set of associated matrices of the non-negative univariate polynomials. This reformulation can be performed since any sum of squares of real polynomials (*sos-polynomials*) or sum of square magnitudes of complex polynomials (*sosm-polynomials*) can be represented by a positive semidefinite matrix. The conic LP deduced in [4] can be generalized as a convex optimization model over an arbitrary set of sos- or sosm-polynomials which are multivariate or univariate polynomials. In this paper, we propose such a model of conic LP:

$$\begin{array}{ll} \text{Find polynomials} & p_1, \ldots, p_m \\ \text{subject to} & \\ & q_i \triangleq \sum_{i=1}^{m} a_{ij} p_j \in \mathcal{K}_i, \ i = 1, \ldots, \mu, \end{array} \tag{1}$$

where $A = (a_{ij})$ is a $(\mu \times m)-$matrix and $\mathcal{K}_i$ is one of the cones of polynomials as follows:

- The cone of non-negative polynomials on $\mathbb{R}, [a, b], [0, +\infty) \subset \mathbb{R}$, the unit circle $\mathbb{T} \subset \mathbb{C}$ or the arc

  $$\mathbb{T}_{uv} = \{z \in \mathbb{T} : \ \arg(u) \leq \arg(z) \leq \arg(v)\},$$

  where $u, v \in \mathbb{T}, \ 0 < \arg(v) - \arg(u) < 2\pi$;
- The cone of sums of squares of real-coefficient polynomials on $\mathbb{R}^n$;
- The cone of sums of square magnitudes of complex-coefficient polynomials on the $n-$torus $\mathbb{T}^n$.

The feasibility part of the next problem is an example of the model (1):

$$\begin{aligned} &\text{minimize} \quad \delta \in (\delta_*, \delta^*) \subset \mathbb{R}_+ \\ &\text{subject to} \\ &\qquad\qquad (p_1, \dots, p_m) \in \mathcal{K}, \\ &\qquad\qquad q_i \triangleq \sum_{i=1}^{m} a_{ij}(\delta) p_j \in \mathcal{K}_i, i = 1, \dots, \mu, \end{aligned} \tag{2}$$

where $A(\delta) = (a_{ij}(\delta))_{i,j}$ is a $(\mu \times m)-$matrix whose entries are real univariate polynomials in δ, $\mathcal{K}_i$ is one of the cones of polynomials described above, and $\mathcal{K}$ is a Cartesian product of some real vector spaces of real or complex polynomials. The problem (2) generalizes the one proposed in [4]. It can also be solved by a combination of a bisection rule on the parameter δ and a feasibility problem as (1).

This paper is organized as follows. Some results on non-negative univariate polynomials on the real line and on the unit circle are presented in Section 2. Sections 3 and 4 summarize the results for sos- and sosm-polynomials. All these sections show how the coefficients of sos(m)-polynomials linearly depend on the entries of their associated Gram matrices. This dependence is the key to reformulate the problem (1) as a conic LP which is presented in Section 6. The next three sections illustrate three examples of low-pass filter design problems with infinite (or finite) impulse response (IIR or FIR). Section 10 states the conclusions.

2. Non-negative univariate polynomials

In this section, we summarize some important results for non-negative univariate polynomials defined on the real line or on the unit circle in the complex plane. In particular, it is shown that the coefficients of such polynomials linearly depend on the entries of their Gram matrices.

2.1. On the real line

It is known that any non-negative real-coefficient polynomial in one variable in $\mathbb{R}$ is a sum of at most 2 squares [6, 7]. Given a real polynomial

$$p(x) = \sum_{i=0}^{2d} p_i x^i \in \mathbb{R}[x]$$

of degree $2d$. Let us consider the sequence $H_i^{(d+1)} \in \mathbb{R}^{(d+1)\times(d+1)}$, $i = 0, 1, \dots, d$ of Hankel matrices in which the first column vector is the i-th identity vector e_i of length $d+1$, i.e., e_i has zero elements except for a 1 at position with index i (where we start counting from zero). It is easy to show now that the polynomial p can be represented by the $(d+1) \times (d+1)$ matrix P as follows

$$p(x) = \mathbf{v}_d(x)^T P \mathbf{v}_d(x) \tag{3}$$

with

$$\mathbf{v}_d(x) = [1 \quad x \quad \dots \quad x^d]^T$$

and

$$p_i = \mathrm{Trace}(H_i^{(d+1)} P), \quad i = 0, 1, \dots, 2d.$$

Proposition 1. (see also [8, Theorem 2.7]) *The polynomial p is sos if and only if p can be represented as (3) with the matrix P positive semidefinite.*

In the following we will see that the coefficients of a non-negative polynomial on the closed interval $[a, b]$ and the infinite interval $[0, +\infty)$ in $\mathbb{R}$ are linearly depending on the coefficients of two polynomials which are non-negative on the whole real line $\mathbb{R}$. By Proposition 1, it is clear that the coefficients of the initial polynomial are linear combinations of the entries of two positive semidefinite matrices. From the Markov-Lukács Theorem [9, 8], we have the following two propositions.

Proposition 2. *Let $p(x) \in \mathbb{R}[x]$ be a univariate polynomial of degree d and let $d_1 = \lfloor \frac{d}{2} \rfloor$.*

i) *If d is even then $p(x)$ is non-negative on the interval $[a, b] \subset \mathbb{R}$ if and only if there exist two polynomials $p_1(x), p_2(x)$ of degree $d_1, d_1 - 1$, respectively, such that*

$$\begin{pmatrix} p_0^0 \\ p_1^0 \\ p_2^0 \\ \vdots \\ p_{d-2}^0 \\ p_{d-1}^0 \\ p_d^0 \end{pmatrix} = \underbrace{\begin{pmatrix} p_0^1 & 0 & 0 & p_0^2 \\ p_1^1 & 0 & p_0^2 & p_2^2 \\ p_2^1 & p_0^2 & p_1^2 & p_3^2 \\ \vdots & \vdots & \vdots & \vdots \\ p_{d-2}^1 & p_{d-4}^2 & p_{d-3}^2 & p_{d-2}^2 \\ p_{d-1}^1 & p_{d-3}^2 & p_{d-2}^2 & 0 \\ p_d^1 & p_{d-2}^2 & 0 & 0 \end{pmatrix}}_{\triangleq L_1} \begin{pmatrix} 1 \\ -1 \\ b+a \\ -ab \end{pmatrix}, \tag{4}$$

where $[p_0^0, p_1^0, \dots, p_d^0]^T$, $[p_0^1, \dots, p_d^1]^T$ and $[p_0^2, \dots, p_{d-2}^2]^T$ denote the column vectors of coefficients of the polynomials $p(x)$, $p_1(x)^2$ and $p_2(x)^2$, respectively.

ii) *If d is odd then $p(x)$ is non-negative on the interval $[a,b] \subset \mathbb{R}$ if and only if there exist two polynomials $p_1(x), p_2(x)$ of degree d_1 such that*

$$\begin{pmatrix} p_0^0 \\ p_1^0 \\ p_2^0 \\ \vdots \\ p_{d-2}^0 \\ p_{d-1}^0 \\ p_d^0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & p_0^2 & p_0^1 \\ p_0^1 & p_0^2 & p_1^2 & p_1^1 \\ \vdots & \vdots & \vdots & \vdots \\ p_{d-1}^1 & p_{d-1}^2 & p_d^2 & p_d^1 \\ p_d^1 & p_d^2 & 0 & 0 \end{pmatrix}}_{\triangleq L_2} \begin{pmatrix} 1 \\ -1 \\ b \\ -a \end{pmatrix}, \tag{5}$$

where $[p_0^0, p_1^0, \dots, p_d^0]^T$, $[p_0^1, \dots, p_d^1]^T$ and $[p_0^2, \dots, p_{d-2}^2]^T$ denote the column vectors of coefficients of the polynomials $p(x)$, $p_1(x)^2$ and $p_2(x)^2$, respectively.

Proof. This proposition is a direct consequence of the Markov-Lukács Theorem [9] (see also [8, Theorem 2.9]) which states that a polynomial $p(x)$ is non-negative on the interval $[a,b]$ if and only if there exist two polynomials $p_1(x), p_2(x)$ which are non-negative on $\mathbb{R}$ such that

$$p(x) = \begin{cases} p_1(x)^2 + (x-a)(b-x)p_2(x)^2, \deg p_2 + 1 = \deg p_1 = d_1, & \text{if } d = 2d_1, \\ \\ (x-a)p_1(x)^2 + (b-x)p_2(x)^2, \deg p_1 = \deg p_2 = d_1, & \text{if } d = 2d_1 + 1. \end{cases}$$

$\square$

We finish this subsection by discussing the semi-infinite interval $[0, +\infty)$.

Proposition 3. *The polynomial $p(x)$ of degree d is non-negative on the semi-infinite interval $[0, +\infty)$ if and only if there exist two non-negative polynomials $p_1(x), p_2(x)$ of degree $2d_1 = 2\lfloor \frac{d}{2} \rfloor, 2d_2 = 2\lfloor \frac{d-1}{2} \rfloor$, respectively, such that the coefficients of p, p_1, p_2 satisfy*

$$\begin{pmatrix} p_0^0 \\ p_1^0 \\ \vdots \\ p_{d-1}^0 \\ p_d^0 \end{pmatrix} = \underbrace{\begin{pmatrix} p_0^1 & 0 \\ p_1^1 & p_0^2 \\ \vdots & \vdots \\ p_{2d_1}^1 & p_{2d_1}^2 \\ 0 & p_{2d_1+1}^2 \end{pmatrix}}_{\triangleq L_3} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{6}$$

if d is odd, or

$$\begin{pmatrix} p_0^0 \\ p_1^0 \\ \vdots \\ p_{d-1}^0 \\ p_d^0 \end{pmatrix} = \underbrace{\begin{pmatrix} p_0^1 & 0 \\ p_1^1 & p_0^2 \\ \vdots & \vdots \\ p_{d-1}^1 & p_{d-2}^2 \\ p_d & 0 \end{pmatrix}}_{\triangleq L_4} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{7}$$

if d is even.

Proof. This is also a consequence of [10, Theorem 5.1]. A polynomial $p(x) \in \mathbb{R}[x]$ is non-negative on $[0, +\infty)$ if and only if two polynomials $p_1(x), p_2(x)$ of degree d_1, d_2, respectively are non-negative on $\mathbb{R}$ such that

$$p(x) = p_1(x) + xp_2(x).$$

$\square$

2.2. On the unit circle in the complex plane

Let

$$p(z) = \frac{1}{2}\left(\sum_{k=0}^{d}(a_k + \imath b_k)z^{-k} + \sum_{k=0}^{d}(a_k - \imath b_k)z^k\right). \tag{8}$$

be a complex Laurent polynomial of degree d, where $\imath = \sqrt{-1}, b_0 = 0$. It follows from the Fejér-Riesz Theorem that any complex Laurent polynomial $p(z)$ non-negative on the unit circle $\mathbb{T}$ is always a square magnitude of a complex polynomial. That is, $p(z) \geq 0, \forall z \in \mathbb{T}$ if and only if $p(z) = |q(z)|^2, \forall z \in \mathbb{T}$ for some $q(z) \in \mathbb{C}[z]$. In (8), if set $p_k = a_k - \imath b_k, k = 0, \ldots, d$, then

$$p(z) = \mathrm{Re}(\sum_{k=0}^{d} p_k z^k) = a_0 + \sum_{k=1}^{d}(a_k \cos k\theta + \imath b_k \sin k\theta), \quad z = e^{\imath\theta}, \theta \in [-\pi, \pi].$$

Given two complex numbers $u, v \in \mathbb{T}$ satisfying $0 < \arg(v) - \arg(u) < 2\pi$. Set $\omega_u = \arg(u)$. The following two propositions show us that the coefficients of a trigonometric polynomial which is non-negative on $\mathbb{T}$ or T_{uv} depend linearly on the entries of some positive semidefinite Hermitian matrices.

Proposition 4. [4] *The trigonometric polynomial (8) of degree d is non-negative on the unit circle if and only if there exists an $X \in \mathbb{H}_+^{d+1}$ where $\mathbb{H}_+^{d+1}$ is defined as the* *set of Hermitian positive semidefinite matrices of order $d+1$, such that*

$$a_k + \imath b_k = \mathrm{Trace}(T_{k+1}^{(d+1)} X), \quad k = 0, 1, \ldots, d, \tag{9}$$

where $T_{k+1}^{(d+1)}$ denotes the $(d+1) \times (d+1)$ Toeplitz matrix whose entries are zero except for some 2's at the $(i,j)-th$ positions such that $i - j = k+1$.

Proposition 5. *The polynomial $p(z)$ as in (8) is non-negative on the arc $\mathbb{T}_{uv}$ if and only if there exist two polynomials $p_1(z), p_2(z)$ of the form (8) with $\deg(p_1) = d, \deg(p_2) = d-1$, which are non-negative on $\mathbb{T}$, such that*

$$\begin{pmatrix} p_0^0 \\ p_1^0 \\ p_2^0 \\ \vdots \\ p_{d-2}^0 \\ p_{d-1}^0 \\ p_d^0 \end{pmatrix} = \underbrace{\begin{pmatrix} p_0^1 & p_0^2 & 0 & p_0^2 \\ p_1^1 & p_1^2 & p_0^2 & p_2^2 \\ p_2^1 & p_2^2 & p_1^2 & p_3^2 \\ \vdots & \vdots & \vdots & \vdots \\ p_{d-2}^1 & p_{d-2}^2 & p_{d-3}^2 & p_{d-1}^2 \\ p_{d-1}^1 & p_{d-1}^2 & p_{d-2}^2 & 0 \\ p_d^1 & 0 & p_{d-1}^2 & 0 \end{pmatrix}}_{\triangleq M} \begin{pmatrix} 1 \\ -2\cos(\frac{\omega_v - \omega_u}{2}) \\ e^{-\imath \frac{\omega_v + \omega_u}{2}} \\ e^{\imath \frac{\omega_v + \omega_u}{2}} \end{pmatrix}, \tag{10}$$

where $[p_0^0, p_1^0, \ldots, p_d^0]^T$, $[p_0^1, \ldots, p_d^1]^T$ and $[p_0^2, \ldots, p_{d-1}^2]^T$ denote the column vectors of coefficients of the polynomials p, p_1 and p_2, respectively.

Proof. One can see in [8, Theorem 6.12] that a trigonometric polynomial $p(z)$ of degree d is non-negative on the arc T_{uv} if and only if there exist two non-negative trigonometric polynomials $p_1(z), p_2(z)$ of degree d and $d-1$, respectively, such that

$$p(z) = p_1(z) + \left(e^{-\imath \frac{\omega_v + \omega_u}{2}} z + e^{\imath \frac{\omega_v + \omega_u}{2}} z^{-1} - 2\cos\frac{\omega_v - \omega_u}{2}\right) p_2(z).$$

Equating the coefficients of the polynomials on the both sides concludes the proof. □

3. Sums of squares of multivariate real polynomials

Let n, d be two natural numbers greater than 1. Let $\Omega(n,d)$ be the set of all possible exponents of the monomials in n variables of degree less than or equal to d. In this paper, "degree" stands for "total degree". Let

$$\Gamma(n,d) = \Omega(n,d) + \Omega(n,d) = \{\alpha + \beta : \alpha, \beta \in \Omega(n,d)\}. \tag{11}$$

Let $\Sigma(n, 2d)$ be the subset of $\mathbb{R}[x]_{n,2d}$ consisting of the real polynomials in n variables of degree $2d$ which are sums of squares. Then $f(x) \in \Sigma(n, 2d)$ if and only if it can be written as [7, 11, 12]

$$f(x) = \sum_{\gamma \in \Gamma(n,d)} f_\gamma x^\gamma = \mathbf{v}_{2d}(x)^T \mathbf{f} = \sum_{j=1}^{r} h_j(x)^2 = \mathbf{v}_d(x)^T (HH^T) \mathbf{v}_d(x), \qquad (12)$$

where $\mathbf{v}_d(x)$ denotes the column vector of monomials x^γ of degree not greater than d arranged in a given order, $\mathbf{x}^T$ denotes the transposed vector of $\mathbf{x}$, $\mathbf{f}$ is the column vector of the corresponding coefficients f_γ and H is the matrix whose jth column is the vector of coefficients $\mathbf{h}_j$ of the polynomial $h_j(x)$ corresponding to the ordering in the vector $\mathbf{v}_d(x)$. Note that the symmetric matrix HH^T of order $d+1$ satisfying (12) is positive semidefinite. Furthermore, the coefficients of f can be represented as a linear combination of the entries of the matrix HH^T.

Proposition 6. *If $f(x)$ satisfies (12) then*

$$\sum_{\substack{\alpha+\beta=\gamma \\ \alpha,\beta \in \Omega(n,d)}} \left(\sum_{k=1}^{r} h_{k\alpha} h_{k\beta} \right) = f_\gamma, \qquad \forall \gamma \in \Gamma(n, d), \qquad (13)$$

where the $(\alpha, \beta)-$th position of the matrix HH^T is $\sum_{k=1}^{r} h_{k\alpha} h_{k\beta}$.

The linear dependency of the coefficients f_γ on the entries of the matrix HH^T is as follows.

Proposition 7. [13] *The real $n-$variable polynomial $f(x)$ of degree $2d$ is a sum of squares if and only if there exists a real positive semidefinite symmetric matrix X of order $d+1$ with the entries $X_{\alpha\beta}, \alpha, \beta \in \Omega(n, d)$ such that*

$$\sum_{\alpha+\beta=\gamma} X_{\alpha\beta} = f_\gamma, \qquad \forall \gamma \in \Gamma(n, d). \qquad (14)$$

4. Sums of square magnitudes of multivariate complex polynomials

A complex Laurent polynomial which is of the form

$$p(z) = \sum_{k=1}^{r} |q_k(z)|^2, \forall z \in \mathbb{T}^n, \qquad (15)$$

where

$$q_k(z) = \sum_{\alpha\in\Omega(n,d)} q_{k\alpha} z^\alpha \in \mathbb{C}[z], \forall k = 1, \ldots, m,$$

is called a sum of square magnitudes of polynomials (sosm-polynomial). Denote by $\Sigma^{\Im}(n, d)$ the set of all sosm-polynomials in n variables and of degree d. Then

$$\Gamma^{\Im}(n, d) = \Omega(n, d) - \Omega(n, d) = \{\alpha - \beta : \ \alpha, \beta \in \Omega(n, d)\}. \tag{16}$$

We also assume that this set and $\Omega(n, d) \subset \Gamma^{\Im}(n, d)$ are endowed with a monomial ordering " $\leqq$ ". The polynomial $p(z) \in \Sigma^{\Im}(n, d)$ can be expressed as

$$p(z) = \sum_{k=1}^{r} |q_k(z)|^2 = \mathbf{v}_d(z)^H \Big(\sum_{k=1}^{r} \bar{\mathbf{q}}_k \mathbf{q}_k^T\Big) \mathbf{v}_d(z) = \mathbf{v}_d(z)^H (\overline{G} G^T) \mathbf{v}_d(z), \quad \forall z \in \mathbb{T}^n, \tag{17}$$

where $\mathbf{q}_k$ denotes the column vectors of the polynomial $q_k(z) \in \mathbb{C}[z_1, \ldots, z_n]$, $G = [\mathbf{q}_1, \ldots, \mathbf{q}_r]$ and $\overline{G}$ is the element-wise conjugate of G. It is clear that $\overline{G}G^T$ is Hermitian and positive semidefinite. Analogously to the case of real multivariate polynomials, we have the following relation.

Proposition 8. *Suppose the matrix $\overline{G}G^T$ has the entries $(\overline{G}G^T)_{\alpha\beta}, \alpha, \beta \in \Omega(n, d)$ then $(\overline{G}G^T)_{\alpha\beta} = \sum_{k=1}^{r} \bar{q}_{k\alpha} q_{k\beta}$. Moreover,*

$$\sum_{\substack{\beta-\alpha=\gamma \\ \alpha,\beta\in\Omega(n,d)}} (\overline{G}G^T)_{\alpha\beta} = p_\gamma, \ \forall \gamma \in \Gamma^{\Im}(n, d), \gamma \geqq 0. \tag{18}$$

5. A perturbation of the associated matrices of polynomials

In this section, we concentrate on the following perturbation

$$X + sI,$$

where X is a square matrix, I denotes the identity matrix of the same order as X and $s \in \mathbb{R}$. This perturbation will be used to model problem (1) as a conic LP. We have seen in the previous sections that each non-negative polynomial is always corresponding to some real or complex positive semidefinite Hermitian matrix. In this section, we present some propositions on the relationship between the polynomials corresponding to the perturbed matrices. That is, assume X_1, X_2 are the associated matrices of the polynomial p and $X_1 + s_1 I, X_2 + s_2 I$ are the associated matrices of the polynomial $\tilde{p}$, we then demonstrate the relationship between the coefficients of p

and $\tilde{p}$. These propositions are useful in analyzing the properties of the conic LP (19), which is presented in Section 6, for some particular cases. It should be noted that an associated matrix of a polynomial does not need to be positive semidefinite but its entries and the coefficients of the corresponding polynomial are linearly dependent as described in the previous sections. The following six propositions are direct consequences of Propositions 1-5.

Proposition 9. *Suppose* $X, X+sI, s \in \mathbb{R}$ *are associated matrices of the polynomials* $p(x), \tilde{p}(x)$ *defined on the real line,* $\deg(p) = \deg(\tilde{p}) = 2d$. *Then*

$$\tilde{p}(x) = p(x) + s\sum_{k=0}^{d} x^{2k}.$$

Proposition 10. *Suppose* $(X_1, X_2), (X_1 + s_1 I, X_2 + s_2 I), s \in \mathbb{R}$ *are associated matrices of the polynomials* $p(x), \tilde{p}(x)$, *respectively, defined on the interval* $[a, b] \subset \mathbb{R}$.

i) *If* $\deg(p) = 2d$ *then*

$$\tilde{p}(x) = p(x) + s_1\sum_{k=0}^{d} x^{2k} + s_2(x-a)(x-b)\sum_{k=0}^{d-1} x^{2k};$$

ii) *If* $\deg(p) = 2d+1$ *then*

$$\tilde{p}(x) = p(x) + s_1(x-a)\sum_{k=0}^{d} x^{2k} + s_2(b-x)\sum_{k=0}^{d} x^{2k}.$$

Proposition 11. *Suppose two polynomials* $p(x), \tilde{p}(x)$, $\deg(p) = \deg(\tilde{p}) = d$ *defined on the semi-infinite interval* $[0, +\infty)$ *have associated matrices* $(X_1, X_2), (X_1 + s_1 I, X_2 + s_2 I), s \in \mathbb{R}$, *respectively. Then*

$$\tilde{p}(x) = p(x) + s_1\sum_{k=0}^{\lfloor d/2 \rfloor} x^{2k} + s_2 x\sum_{k=0}^{\lfloor (d-1)/2 \rfloor} x^{2k}.$$

Proposition 12. *Suppose* $p(x), \tilde{p}(x) \in \mathbb{R}[x]_{n,2d}$ *are associated to the matrices* $X, X + sI$, *respectively. Then*

$$\tilde{p}(x) = p(x) + s\sum_{\alpha \in \Omega(n,d)} x^{2\alpha}.$$

Proposition 13. [4, Proposition 4] *Let $p_1(z)$, $p_2(z)$ and $p_3(z)$ be the trigonometric polynomials of degree d on $\mathbb{T}$, $\mathbb{T}_u$, $\mathbb{T}_{\overline{u}}$, respectively and $X_1, (X_{21}, X_{22}), (X_{31}, X_{32})$ be, respectively, their corresponding matrices. For some real numbers $s, s_{21}, s_{22}, s_{31}, s_{32}$, let $\tilde{p}_1(z)$, $\tilde{p}_2(z)$ and $\tilde{p}_3(z)$ denote the polynomials corresponding to $X_1 + sI, (X_{21} + s_{21}I, X_{22} + s_{22}I), (X_{31} + s_{31}I, X_{32} + s_{32}I)$, respectively. Then*

$$\begin{aligned}
&\tilde{p}_1(z) = p_1(z) + (d+1)s, \quad z \in \mathbb{T},\\
&\tilde{p}_2(z) = p_2(z) + (d+1)s_{21} + 2ds_{22}(\cos\omega_z - \cos\omega_u), \quad z = e^{j\omega_z} \in \mathbb{T}_u,\\
&\tilde{p}_3(z) = p_3(z) + (d+1)s_{31} + 2ds_{32}(\cos\omega_u - \cos\omega_z), \quad z = e^{j\omega_z} \in \mathbb{T}_{\overline{u}}.
\end{aligned}$$

Proposition 14. *Suppose $p(z), \tilde{p}(z) \in \mathbb{R}[z_1, \ldots, z_n]$ are multivariate complex Laurent polynomials of degree d with the associate matrices $X, X + sI$, respectively. Then*

$$\tilde{p}(z) = p(z) + \hat{e}s.$$

6. A conic linear programming version

We now return to the problem (1). From the Gram-matrix representation of the polynomials, one can see that each polynomial $q_i \in \mathcal{K}_i, i = 1, \ldots, \mu$ is associated to one or two real or complex Hermitian matrices $Q_{il_i}, l_i \in J_i$, where J_i has one or two elements depending on the type of q_i. This allows us to convert the problem (1) into a problem on positive semidefinite matrices. In this sense, the problem (1) is reformulated as an optimization problem over the perturbed matrices of the associated matrices of polynomials as follows:

$$\begin{aligned}
\text{minimize} \quad & s \qquad\qquad (19)\\
\text{subject to} \quad & \\
& (p_1, \ldots, p_m) \in \mathcal{K},\\
& Q_{il_i} + sI_{il_i} \succeq 0,\ l_i \in J_i, i = 1, \ldots, \mu,\\
& [\mathbf{q}_1^T \ldots \mathbf{q}_\mu^T]^T = \hat{A}(\delta) \cdot [\mathbf{p}_1^T \ldots \mathbf{p}_m^T]^T,
\end{aligned}$$

where J_i has one or two elements depending upon the type of the corresponding polynomial q_i, and

$$\hat{A}(\delta) = \begin{pmatrix} a_{11}(\delta)I & \ldots & a_{1m}(\delta)I \\ a_{21}(\delta)I & \ldots & a_{2m}(\delta)I \\ \ldots & \ddots & \ldots \\ a_{\mu}(\delta)I & \ldots & a_{\mu m}(\delta)I \end{pmatrix}.$$

In problem (19), the matrices $Q_{il_i}, l_i \in J_i$ are corresponding to the polynomial $q_i, i = 1 \ldots, \mu$. It should be noted that if Q_{il_i}, q_i satisfy a feasible point of (19) then q_i is not necessarily belonging to the corresponding cones $\mathcal{K}_i$. However, the coefficients of the polynomials q_i are linearly dependent on the entries of their corresponding matrices because of Propositions 1−8. We call these matrices $Q_{il_i}, i = 1, \ldots, \mu, l_i \in J_i$ the *associated matrices* of the polynomial q_i. Moreover, given a list of matrices $Q_{il_i}, i = 1, \ldots, \mu, l_i \in J_i$, one can compute the corresponding polynomials. In the opposite direction, given a non-negative polynomial, its associated matrices can be derived as follows. It suffices to prove this for the cases of polynomials on the whole real line, the unit circle in the complex plane, and the sos- and sosm-polynomials. An associated matrix X of the polynomial p on the real line (unit circle) is Hankel (Toeplitz), and can be taken as

$$X_{ij} = \frac{p_k}{k+1}, i + j = k, k = 0, \ldots, \deg(p),$$

resp.,

$$X_{ij} = \begin{cases} p_0 & \text{if} \quad i = j \\ \frac{p_k}{2(\deg(p)-k+1)} & \text{if} \quad i - j = k, k = 1, \ldots, \deg(p). \end{cases}$$

The reader is referred to [8] for the general representation of the associated matrices in these cases. For the case of sos-polynomials, the system (13) has $|\Gamma(n, d)|$ equations and $\frac{|\Omega(n,d)|(|\Omega(n,d)|+1)}{2}$ unknowns and it follows from the work by Barvinok [14] that a positive semidefinite matrix which is associated to the polynomial exists. A similar proof for the existence of the positive semidefinite matrices associated to the sosm-polynomials can be found in [15].

We now prove that the problem (19) is a conic LP. We first mention a relation between the positive semidefiniteness of complex and real Hermitian matrices.

Proposition 15. (See also [16, Exercise 4.42]) *A complex Hermitian matrix $X \in \mathbb{H}^\mu$ is positive semidefinite if and only if the matrix*

$$\operatorname{sym}(X) \triangleq \begin{pmatrix} \operatorname{Re}(X) & -\operatorname{Im}(X) \\ \operatorname{Im}(X) & \operatorname{Re}(X) \end{pmatrix},$$

where $\operatorname{Re}(X), \operatorname{Im}(X)$ *are the element-wise real and imaginary parts of* X, *respectively, is positive semidefinite.*

Proposition 16. i) *The cone $\mathbb{S}^\mu_+$ of positive semidefinite real matrices can be viewed as a cone in $\mathbb{R}^{\frac{\mu(\mu+1)}{2}}$;*

ii) *The cone* $\mathbb{H}^{\mu}_{+}$ *of complex positive semidefinite Hermitian matrices can be viewed as a cone in* $\mathbb{R}^{\mu^2}$.

Proof. i) The proof of this part comes from [17, Section 7.2]. The bijection is defined as follows. Suppose the Hilbert space $\mathbb{S}^{\mu}$ is endowed with the standard inner product, i.e., $\langle X, Y\rangle_{\mathbb{S}^{\mu}} = \text{Trace}(XY^T)$. The bijection equals $X = (x_{ij}) \mapsto \tilde{x} = [x_{11}, \ldots, x_{1\mu}, \ldots, x_{\mu\mu}]^T$. The inner product on $\mathbb{R}^{\frac{\mu(\mu+1)}{2}}$ is defined as

$$\langle \tilde{x}, \tilde{y}\rangle_D = \tilde{x}^T D\tilde{y}, \forall \tilde{x}, \tilde{y} \in \mathbb{R}^{\mu(\mu+1)/2},$$

where $D = \text{diag}(d_{11}, \ldots, d_{1\mu}, \ldots, d_{\mu\mu})$ is a diagonal matrix such that $d_{ii} = 1$ and $d_{ij} = 2$ for $1 \leq i < j \leq n$.

ii) The Hermitian matrix space is a Hilbert space with the standard inner product $\langle A, B\rangle_{\mathbb{H}^{\mu}} = \text{Trace}(AB^H)$. It follows from Proposition 15 that $\mathbb{H}^{\mu}$ can be viewed as a subset of $\mathbb{S}^{\mu} \times \mathbb{S}^{\mu}$ endowed with the inner product defined as follows. For all $A, B \in \mathbb{H}^{\mu}$, $A = X + \imath Y, B = U + \imath V, X, U \in \mathbb{S}^{\mu}, Y^T = -Y, V^T = -V, \imath^2 = -1$ we have

$$\begin{aligned}
\langle A, B\rangle_{\mathbb{H}^{\mu}} &= \frac{1}{2}\text{Trace}(\text{sym}(A)\text{sym}(B)^T) = \langle X, U\rangle_{\mathbb{S}^{\mu}} + \langle Y, V\rangle_{\mathbb{S}^{\mu}} \\
&= \langle \tilde{x}, \tilde{u}\rangle_D + \langle \tilde{y}, \tilde{v}\rangle_D.
\end{aligned}$$

The Hermitian matrix space is hence bijective with $\mathbb{R}^{\mu^2}$. □

In fact, Hill and Water [18] proved that the $\mathbb{R}-$ Hilbert space $\mathbb{H}^{\mu}$ with the inner product $\langle ., .\rangle_{\mathbb{H}^{\mu}}$ is isometrically isomorphic to the $\mathbb{R}-$ Hilbert space $\mathbb{R}^{\mu^2}$ with the standard inner product. The isomorphism in this case is given by

$$A = (a_{kl}) \longmapsto \left(a_{11}, \sqrt{2}\text{Re}(a_{12}), \sqrt{2}\text{Im}(a_{12}), \ldots, a_{22}, \sqrt{2}\text{Re}(a_{23}), \ldots, a_{\mu\mu}\right).$$

Theorem 1. *The problem (19) is a conic LP.*

Proof. Each Hermitian matrix $Q_{il_i},\ i = 1, \ldots, \mu,\ l_i \in J_i$ in the problem (19) is relaxed as a vector $\tilde{q}_{il_i}$ defined in the proof of the Proposition 16. So, the problem (19) can be reformulated as a problem over the set of all lists

$$\tilde{x} = [s, \tilde{q}_{1l_1}, \ldots, \tilde{q}_{\mu l_\mu}, \mathbf{p}_1^T, \ldots, \mathbf{p}_m^T]^T \in \mathbb{R}^{\eta}, \tag{20}$$

for a suitable natural number η. Moreover, the set of all lists of the form (20) is a convex cone. Indeed, if

$$\tilde{x} = [s, \tilde{q}_{1l_1}, \ldots, \tilde{q}_{\mu l_\mu}, \mathbf{p}_1^T, \ldots, \mathbf{p}_m^T]^T, \quad \tilde{y} = [t, \tilde{v}_{1l_1}, \ldots, \tilde{v}_{\mu l_\mu}, \mathbf{u}_1^T, \ldots, \mathbf{u}_m^T]^T$$

belong to this set and $\xi, \tau \in \mathbb{R}_+$ then

- the polynomials $\xi q_i + \tau v_i, i = 1, \ldots, \mu$ are linear combinations of $\xi p_j + \tau u_j, j = 1, \ldots, m$ as in (19);
- $\xi Q_{il_i} + \tau V_{il_i}$ is associated to $\xi q_i + \tau v_i$;
- $\xi Q_{il_i} + \tau V_{il_i} + (\xi s + \tau t)I$ is positive semidefinite.

Finally, by substituting the corresponding relaxed vectors $\tilde{q}_{il_i}$ of the polynomial $q_i, i = 1, \ldots, \mu$ into the equality $q_i = \sum_{k=1}^m a_{ij} p_j$, we get a system of homogeneous linear equations in $\tilde{x}$. $\square$

The problem (19) can be viewed as the following polynomial optimization one described in [19, 20]:

$$\min_q f(q) \text{ subject to } q \in \mathbf{K}, \tag{21}$$

where $f : \mathbb{R}^k \to \mathbb{R}$ and $\mathbf{K}$ is a subset (not necessarily a cone but usually a compact set defined by polynomial equalities and polynomial inequalities) in $\mathbb{R}^k$. In [19, 20], the authors prove that this can be solved by using its equivalent *generalized problem of moments*. In our situation, one can take $\mathbf{K}$ to be the intersection of the convex cone of all the lists $\tilde{x}$ in (20) and the hyperplanes defined by the corresponding system of homogeneous linear equations. From the theoretical point of view, Problem (19) can be solved by using GloptiPoly [21]. However, there is currently no support in GloptiPoly to handle the set $\mathbf{K}$ in our sense. More precisely, the Löwner inequalities in (19) are equivalent to the polynomial inequalities derived by their principle minors. It is, however, not easy to describe these polynomial inequalities in GloptiPoly.

7. An optimization problem model combining a bisection rule and a conic LP

This section presents an algorithm for solving problem (2). This is a generalization of the algorithm proposed in [4]. We emphasize that the whole problem (2) is not convex in general, but for each $\delta \in (a, b)$ the feasibility problem is of the form (19) and can be solved by SDP solvers. This model can be used to solve the filter design problems presented in the next sections. We will see later, when solving these filter design problems, that our method appears "dual" to the method developed in

[3]. Indeed, their method solves the FIR lowpass filter design problems in variable δ with respect to the sampled values of the frequency while our method solves such a problem in frequency variable with respect to each value of δ. As a generalization of the algorithm in [4], the following algorithm is proposed for solving problem (2).

Algorithm 1.

Input:

- a value $\delta_0 \in (\delta_*, \delta^*)$, should be taken close to δ^*. That is, it should start with a δ_0 so that the optimal value of the corresponding conic LP (19) is negative: $s(\delta_0) < 0$.
- m, the number of the polynomials p_j needed to find and their type, that is, the vector space of polynomials they belong to;
- polynomial matrix $A(\delta) \in \mathbb{R}[\delta]^{\mu \times m}$ and the type of the polynomials q_i;
- a precision $\epsilon > 0$.

Output:

- a minimal value $\delta_{min} \in (\delta_*, \delta_0]$;
- m polynomials p_j satisfying (2).

Initialization: Set $\delta^0_{low} = \delta_*, \delta^0_{up} = \delta_0$ and $\delta^1 = \frac{\delta_{low}+\delta_{up}}{2}$.

At iteration $k \geq 0$:

Set $\delta^{k+1} = \frac{\delta^k_{up}+\delta^k_{low}}{2}$.

While $(\delta^k_{up} - \delta^k_{low})/\delta^k_{up} > \epsilon$ do

1. Solve the convex optimization problem (19) with the input data.
   Let $s^{k+1} = s(\delta^{k+1})$ be the corresponding optimal value.
2. If $s^{k+1} > 0$ then set $\delta^{k+1}_{low} = \delta^{k+1}$, $\delta^{k+1}_{up} = \delta^k_{up}$.
   Else, set $\delta^{k+1}_{up} = \delta^{k+1}$, $\delta^{k+1}_{low} = \delta^k_{low}$.
3. Go to iteration $k + 1$.

The value δ_{min} is the first δ^k satisfying $(\delta^k_{up} - \delta^k_{low})/\delta^k_{up} \leq \epsilon$.

The next two sections present some numerical examples linked to the optimization models (2) and (1). It turns out that both the IIR and FIR low-pass filter design problems can be formulated as the optimization model (2). All experiments illustrated below are implemented in MATLAB 2012a, using the `CVX` Toolbox and were performed on an Intel® Dual Core™@1.85 GHz.

8. IIR low-pass filter

An IIR low-pass filter is a discrete-time filter whose input signal y and output signal w satisfy the equality

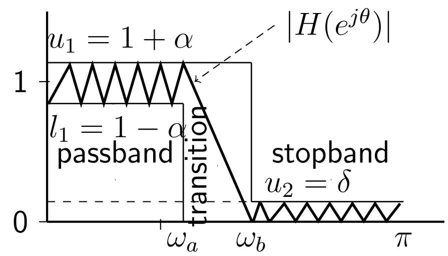
$$\sum_{k=0}^{d} a_k w[t-k] = \sum_{k=0}^{d} b_k y[t-k],$$

with real filter coefficients $\{a_k\}_{k=0}^d, \{b_k\}_{k=0}^d$ and where $y[i], w[i]$ denote the input, output signal at discrete time i, respectively. The corresponding frequency response function for filter degree d, say

$$H(e^{j\theta}) = \frac{\sum_{k=0}^d a_k e^{-jk\theta}}{\sum_{k=0}^d b_k e^{-jk\theta}}, \;\; \theta \in [-\pi, \pi],$$

satisfies the following conditions

$$\begin{array}{ll} |H(e^{\imath\theta})|^2 \geq l_1^2, & \theta \in [0, \omega_a], \\ |H(e^{\imath\theta})|^2 \leq u_1^2, & \theta \in [0, \omega_b], \\ |H(e^{\imath\theta})|^2 \leq \delta^2, & \theta \in [\omega_b, \pi], \\ |H(e^{\imath\theta})|^2 \geq 0, & \theta \in [0, \pi], \end{array} \qquad (22)$$



where $0 < \omega_a < \omega_b < \pi$, $l_1 = 1 - \alpha$, $u_1 = 1 + \alpha$, $0 < \alpha < 1$, $0 < \delta < l_1$.

Three standard filter parameters to be optimized are the stop-band attenuation $u_2 = \delta$, the pass-band ripple $u_1 - l_1 = 2\alpha$ and the filter degree d. There are many methods to solve such problems. In [2], Genin et al. focus on the problem that minimizes the stopband attenuation. We note that a similar method can be used to solve the problem that minimizes the two other filter parameters. In [4], the authors, with the help of the reformulation in [2] that the filter design problem can be cast as an optimization problem over non-negative trigonometric polynomials, reformulate the feasibility problem of the filter design problem above as a conic LP. Moreover, their work, in contrast to [2], concentrates on the problem that minimizes the stop-band attenuation and pass-band ripple simultaneously. Namely, by setting $\alpha = \delta$

and $z = e^{-\imath\theta}$, we minimize the parameter $\delta \in (0,1)$ subjected to the constraints

$$\begin{cases} q_1(z) \triangleq p_1(z) \geq 0, & \arg(z) \in [-\pi, \pi], \\ q_2(z) \triangleq p_1(z) - l_1^2 p_2(z) \geq 0, & \arg(z) \in [0, \omega_a], \\ q_3(z) \triangleq u_1^2 p_2(z) - p_1(z) \geq 0, & \arg(z) \in [0, \omega_b], \\ q_4(z) \triangleq u_2^2 p_2(z) - p_1(z) \geq 0, & \arg(z) \in [\omega_b, \pi], \end{cases} \tag{23}$$

where $l_1^2 = (1-\delta)^2, u_1^2 = (1+\delta)^2, u_2^2 = \delta^2$ and $p_1(z), p_2(z)$ are trigonometric (cosine) polynomials of degree d being the numerator and denominator of $|H(e^{-\imath\theta})|^2$. This problem is of the form (2) with $(\delta_*, \delta^*) = (0,1),\ m = 2,\ \mu = 4,$

$$A(\delta) = \begin{pmatrix} 1 & 0 \\ 1 & -l_1^2 \\ -1 & u_1^2 \\ -1 & u_2^2 \end{pmatrix}$$

and $q_1, \ldots, q_4$ belong to the cones of non-negative polynomials on the unit circle $\mathbb{T}$ in the complex plane or the arcs with the end points $a = e^{\imath\omega_a}, \bar{a} = e^{-\imath\omega_a}, b = e^{\imath\omega_b}, \bar{b} = e^{-\imath\omega_b}$. The solution to this problem can be found in [4]. We now transform this problem to one over the real polynomials defined on $[-1,1]$. The new version satisfies (2) with the feasibility problem as in (1).

IIR low-pass filter design problem on real polynomials. The idea is that each variable z on the unit circle can be viewed as a pair $(x,y) \in [-1,1]^2$ with $x^2 + y^2 = 1$. Suppose $p(z) = \mathrm{Re}(\sum_{k=0}^d p_k z^k) = \sum_{k=0}^d \Big(a_k \cos k\theta + b_k \sin k\theta\Big)$ is a trigonometric polynomial of degree d and $p_k = a_k - \imath b_k, z = e^{\imath\theta} = \cos\theta + \imath \sin\theta \in \mathbb{T}$. We have already known that for each $k \in \mathbb{N}$,

$$\cos(k\theta) = 2^{k-1}\cos^k\theta + k\sum_{r=1}^{\lfloor k/2 \rfloor} \frac{(-1)^r}{r} \binom{k-r-1}{r-1} 2^{k-2r-1}(\cos\theta)^{k-2r}$$

and

$$\sin(k\theta) = \sin\theta\left(\sum_{r=0}^{\lfloor (k-1)/2 \rfloor} (-1)^r \binom{k}{2r+1} (\cos\theta)^{k-2r-1}(1-\cos^2\theta)^r\right),$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and $\lfloor . \rfloor$ denotes the floor function. So, if we set $x = \cos\theta, y =$

$\sin\theta \in [-1,1]$, then the original polynomial can be written

$$p(z) \equiv u(x) + yv(x), \quad \forall (x,y) \in [-1,1]\times[-1,1],\ x^2+y^2=1,$$
$$u, v \in \mathbb{R}[x],\ \deg(u) = \deg(p) = d,\ \deg(v) = d-1.$$

As noticed in [4], the solution polynomials are cosine, i.e., the coefficients b_k in the trigonometric form of $p(z)$ are zero. Hence, there is a one-to-one correspondence between the polynomial $p(z)$ and the polynomial $u(x)$.This allows to write the polynomial $u(x)$ as follows

$$u(x) = a_0 + \sum_{k=1}^{d} a_k \left(2^{k-1}x^k + k \sum_{r=1}^{\lfloor k/2 \rfloor} \frac{(-1)^r}{r} \binom{k-r-1}{r-1} 2^{k-2r-1} x^{k-2r} \right). \tag{24}$$

This means that for a given cosine polynomial p we can find a univariate polynomial $\hat{p}(x) = \hat{p}_0 + \ldots + \hat{p}_d x^d \equiv u(x)$ as in (24). Oppositely, we now show that for a given real polynomial $\hat{p}(x)$ defined on $[-1,1]$, there exists a cosine polynomial p such that $\hat{p}(x) = p(z), x = \cos\theta, z = e^{\imath\theta}, \theta \in [-\pi,\pi]$.

Proposition 17. *Let* $\hat{p}(x) = \hat{p}_0 + \hat{p}_1 x + \ldots + \hat{p}_d x^d \in \mathbb{R}[x], x \in [-1,1]$ *and* $p(z) = \mathrm{Re}(p_0 + p_1 z + \ldots + p_d z^d) = \sum_{k=0}^{d}(a_k \cos k\theta + b_k \sin k\theta), z \in \mathbb{T}, p_k = a_k - jb_k \in \mathbb{C}, \forall k = 1,\ldots,d$ *be two polynomials whose coefficients satisfy the following equalities*

i) $b_l = 0, \forall l = 0,1,\ldots,d,$

ii) $\hat{p}_0 = \sum_{r=0}^{\lfloor d/2 \rfloor} (-1)^r a_{2r},$

iii) $\hat{p}_1 = \sum_{r=1}^{\lfloor d/2 \rfloor} (-1)^r (2r+1) a_{2r+1},$

iv) $\hat{p}_{2j} = 2^{2j-1} a_{2j} + \sum_{l=j+1}^{\lfloor d/2 \rfloor} \left(\frac{(-1)^{l-j} 4^j l(l-j+1)\ldots(l+j-1)}{(2j)!} \right) a_{2l}$
for $j = 1,\ldots,\lfloor d/2 \rfloor$.
Set $k = \lfloor d/2 \rfloor$ *if* d *is odd and* $k = \lfloor d/2 \rfloor - 1$ *if* d *is even. Then*

$$\hat{p}_{2j+1} = 4^j a_{2j+1} + \sum_{l=j+1}^{k} \left(\frac{(-1)^{l-j}}{(2j+1)!} 4^j (2l+1) \prod_{i=l-j+1}^{l+j} i \right) a_{2l+1}$$

for $j = 1,\ldots,k$.

Then $p(z) \geq 0, \forall z \in \mathbb{T}$ *if and only if* $\hat{p}(x) \geq 0, \forall x \in [-1,1]$. *Moreover, let* $\hat{\mathbf{p}} = (\hat{p}_0, \ldots, \hat{p}_d)^T, \mathbf{p} = (a_0, \ldots, a_d)^T$ *then* $\hat{\mathbf{p}} = \Phi \mathbf{p}$ *where*

$$\Phi = \begin{pmatrix} 1 & 0 & -1 & 0 & \ldots & * \\ & 1 & 0 & -3 & \ddots & \vdots \\ & & 2^1 & 0 & \ddots & 0 \\ & & & 2^2 & \ddots & * \\ & & & & \ddots & 0 \\ & & & & & 2^{d-1} \end{pmatrix}.$$

We now return to our low-pass filter design problem. Because of what we have discussed in this section, the filter design problem can be reformulated as a problem on the non-negative real polynomials on intervals in $\mathbb{R}$. That is

$$\begin{aligned} &\text{minimize } \delta \\ &\text{subject to} \\ &\begin{cases} q_1(x) = p_1(x) \geq 0, & \forall x \in [-1,1], \\ q_2(x) = p_1(x) - l_1^2 p_2(x) \geq 0, & \forall x \in [\cos\omega_a, 1], \\ q_3(x) = u_1^2 p_2(x) - p_1(x) \geq 0, & \forall x \in [\cos\omega_b, 1], \\ q_4(x) = u_2^2 p_1(x) - p_1(x) \geq 0, & \forall x \in [-1, \cos\omega_b]. \end{cases} \end{aligned} \tag{25}$$

Analogously to the case of the cosine polynomials, the feasibility problem of (25) can be stated as follows

$$\begin{aligned} &\text{minimize } s \\ &\text{subject to } \; x \in \mathcal{S}, \\ &\begin{pmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \\ \mathbf{q}_3 \\ \mathbf{q}_4 \end{pmatrix} = \begin{pmatrix} I & 0 \\ I & -l_1^2 I \\ -I & u_1^2 I \\ -I & u_2^2 I \end{pmatrix} \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{pmatrix}, \end{aligned} \tag{26}$$

where $\mathcal{S}$ is the set of all the lists (20). We note that each polynomial q_i may have two associated positive semidefinite matrices because of Propositions 2, 3, 5. The problem (26) corresponds to six matrices instead of only five as in the problem over the corresponding trigonometric polynomials.

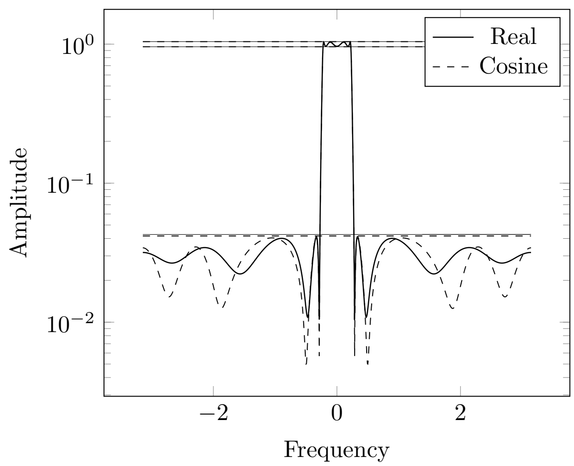


Figure 1: Semilogy graph of IIR amplitude corresponding to real and cosine polynomials with $d = 9, \omega_a = 0.225, \omega_b = 0.275$.

We now perform an example for this case. The input data for Algorithm 1 is taken as

$$d = 9, \omega_a = 0.225, \omega_b = 0.275, \delta_0 = 0.90, \varepsilon = 10^{-3}.$$

Solving the problem (25) with respect to the above data, we get $\delta_{\min}^{real} \simeq 0.0428$, the minimal value of the corresponding feasibility problem at the last iteration is $s(\delta_{\min}^{real}) \simeq -3.6792 \times 10^{-10}$. Two polynomials are found as

$$\begin{aligned} \hat{\mathbf{p}}_1 = \quad & (0.0498 \quad -0.2123 \quad 0.1350 \quad 0.6763 \quad -0.8692 \\ & -1.0743 \quad 2.1534 \quad -0.1190 \quad -1.3335 \quad 0.5938)^T, \\ \hat{\mathbf{p}}_2 = \quad & (23.0312 \quad -69.6572 \quad 52.0457 \quad 11.8250 \quad 10.5988 \\ & -53.0473 \quad 13.5448 \quad 22.6068 \quad -11.9480 \quad 1.0000)^T. \end{aligned}$$

These polynomials are then converted to cosine ones by applying Proposition 17.

To give a comparison between problems over real and cosine polynomials, an experiment directly solving the two problems with respect to the same input data is performed. This experiment gives $\delta_{\min}^{trig} \simeq 0.0417, s(\delta_{\min}^{trig}) \simeq -1.0131 \times 10^{-9}$. For convenience to compare these two experiments, the dependence of the amplitude functions (defined by the cosine and real polynomials) on the frequency variable $\theta \in [-\pi, \pi]$ are both depicted on Figure 1.

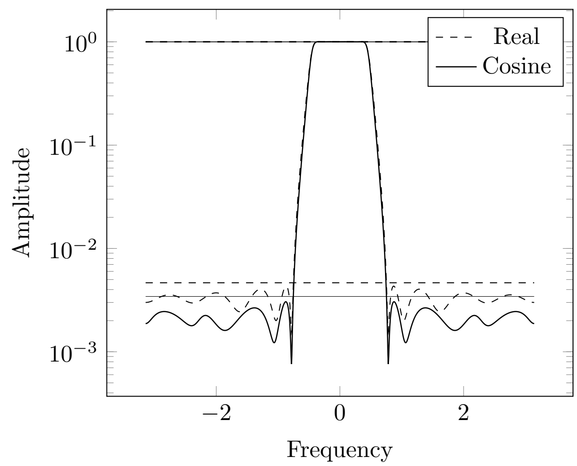


Figure 2: Semilogy graph of IIR amplitude corresponding to real and cosine polynomials with $d = 9$, $\omega_a = 0.12\pi$, $\omega_b = 0.24\pi$.

An analogous plot for the input data

$$d = 9, \omega_a = 0.12\pi, \omega_b = 0.24\pi, \delta_0 = 0.90, \varepsilon = 10^{-3}.$$

is shown in Figure 2. In this situation, one has

$$\delta^{tri}_{\min} \simeq 0.0034, \quad \delta^{real}_{\min} \simeq 0.0047.$$

Even though both problems over the trigonometric and the real polynomials can be solved for the filter degree up to 70 by applying the model (2), it is not convenient to convert the results of these problems into the other. Indeed, the determinant of the matrix Φ is an exponential function in d, the linear system $\mathbf{p} = \Phi^{-1}\hat{\mathbf{p}}$ may lead to an under/overflow in the computations.

Besides the examples described here, we have performed several other tests with different choices for the parameters. It turns out that for the IIR problem over cosine polynomials the degree d can be taken up to 70, whith $\omega_b - \omega_a$ between 0.005 and 0.05 and ω_a from 0.125 up to 3.14. For IIR and FIR problems over real polynomials, with the same possible parameters above, the degree can be up to 30.

Remark 1. If δ is viewed as a variable in $[-1, 1]$ then the above problem can be rewritten as

minimize δ^2

subject to

$$\begin{cases} q_1(x) = p_1(x) \geq 0, & \forall x \in [-1,1], \\ q_2(x,\delta) = p_1(x) - (1-\delta^2)p_2(x) \geq 0, & \forall (x,\delta) \in [Re(a),1] \times [-1,1], \\ q_3(x,\delta) = (1+\delta^2)p_2(x) - p_1(x) \geq 0, & \forall (x,\delta) \in [Re(b),1] \times [-1,1], \\ q_4(x,\delta) = \delta^2 p_1(x) - p_1(x) \geq 0, & \forall (x,\delta) \in [-1,Re(b)] \times [-1,1]. \end{cases} \tag{27}$$

Remark 2. We finish this section by giving a comparison of our model (1) with the "sum of squares program" (SOSP) [22] which is used to solve semialgebraic problems in [23]. If we do not use sum of square magnitudes polynomials, problem (1) could be reformulated in a direct way as a problem solvable by SOSTOOLS.

9. FIR lowpass filter

In this section, we solve the FIR low-pass filter design problem proposed in the paper of Wu et al. [3]. Such a problem minimizes either the pass-band ripple, the stop-band attenuation, or the filter degree so that the frequency response function

$$H(e^{\imath\theta}) = h_0 + h_1 e^{-\imath\theta} + \ldots + h_d e^{-\imath d\theta},\ \theta \in [-\pi,\pi]$$

satisfies the constraints as in (22). Here, since $H(e^{\imath\theta})$ is $2\pi-$periodic and $H(e^{-\imath\theta}) = \overline{H(e^{\imath\theta})}, \forall\theta \in [-\pi,\pi]$, one can consider the problem on $[0,\pi]$ instead of $[-\pi,\pi]$. We now consider the problem that minimizes the parameter stopband attenuation δ and the other parameters (filter degree d, passband ripple α) are fixed. It is written as

$$\begin{array}{ll} \text{minimize} & \delta \in (0,+\infty) \\ \text{subject to} & \\ & \begin{array}{ll} 1/\alpha^2 \leq |H(e^{\imath\theta})|^2 \leq \alpha^2, & \theta \in [0,\omega_a], \\ |H(e^{\imath\theta})|^2 \leq \delta, & \theta \in [0,\omega_b], \\ |H(e^{\imath\theta})|^2 \geq 0, & \theta \in [0,\pi]. \end{array} \end{array}$$

In [3], the authors prove that the above problem (in variables δ,θ) satisfies a semi-infinite optimization problem. That is, the objective function is a univariate convex function in δ and for each $\theta \in [0,\pi]$ all the constraint functions are convex in δ. In fact, the functions in this case are linear in δ for each $\theta \in [0,\pi]$. Each semi-infinite inequality constraint of the problem is then approximated by $N \simeq 15d$ inequalities in δ with respect to N sampling frequencies

$$0 < \theta_1 < \ldots < \theta_N < \pi.$$

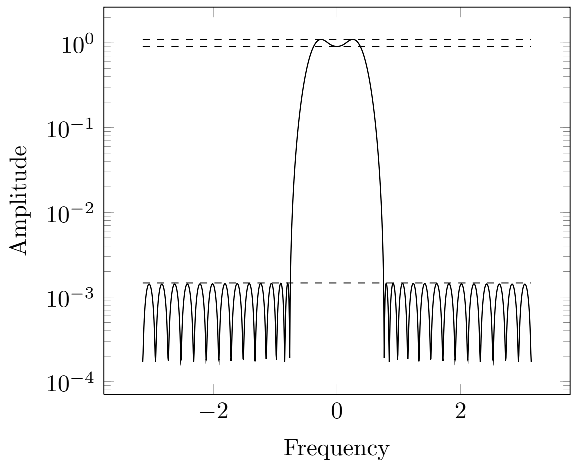


Figure 3: Semilogy graph of FIR amplitude corresponding to $d = 30$, $\omega_a = 0.12\pi$, $\omega_b = 0.24\pi$.

The resulting optimization problem is hence a linear program and can be solved by an SDP solver.

For our method, by setting $p(z) = |H(z)|^2, z = e^{\imath\theta}, \theta \in [-\pi, \pi]$, this problem is rewritten in the form (2) as follows:

$$\begin{array}{lll} \text{minimize} & \delta \in (0, +\infty) & \\ \text{subject to} & & \\ & q_1(z) \triangleq p(z) - 1/\alpha^2 \geq 0, & \arg(z) \in [0, \omega_a], \\ & q_2(z) \triangleq \alpha^2 - p(z) \geq 0, & \arg(z) \in [0, \omega_b], \\ & q_3(z) \triangleq \delta - p(z) \geq 0, & \arg(z) \in [\omega_b, \pi], \\ & q_4(z) \triangleq p(z) \geq 0, & \arg(z) \in [0, \pi]. \end{array} \tag{28}$$

We illustrate this problem for the same input data in [3, Section 4] as follows:

$$d = 30, \quad \epsilon = 10^{-3}, \quad \alpha = 1.1, \quad \omega_a = 0.12\pi, \quad \omega_b = 0.24\pi, \quad \delta_0 = 0.90.$$

Using Algorithm 1 with $\delta_0 = 0.9$, $\epsilon = 10^{-3}$, we get the optimal value $\delta_{min} = 0.1465$ with respect to the optimal value of the feasibility problem $s(\delta_{min}) = -1.0106 \times 10^{-9}$. Figure 3 shows the amplitude of this result.

10. Conclusion

We have proposed an optimization model for finding polynomials such that some specific linear combinations of those polynomials are non-negative or sos(m)-polynomials.

It has been proved that this problem is a conic linear program and can be solved by an SDP solver. We have also generalized an optimization problem that can be solved by combining a bisection rule on an appropriate parameter and a scheme of this model solving the feasibility problem. Some IIR and FIR lowpass filter design problems have been solved by utilizing this model.

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